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# Using the Hadamard and related transforms for simplifying the spectrum of the quantum baker's map

**Arul Lakshminarayan and N Meenakshisundaram**

Department of Physics, Indian Institute of Technology Madras, Chennai, 600 036, India

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**Abstract**

We rationalize the somewhat surprising efficacy of the Hadamard transform in simplifying the eigenstates of the quantum baker's map, a paradigmatic model of quantum chaos. This allows us to construct closely related, but new, transforms that do significantly better, thus nearly solving many states of the quantum baker's map. These transforms, which combine the standard Fourier and Hadamard transforms in an interesting manner, are constructed from eigenvectors of the shift permutation operator that are also simultaneous eigenvectors of bit-flip (parity) and possess bit-reversal (time-reversal) symmetry.

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## 1. Introduction

The eigenstates of quantized chaotic systems continue to be studied vigorously and present several challenges. Exactly or nearly exactly solvable models are desirable in this setting, but are few and far between, especially for systems that display generic behaviour. We had pointed out earlier [1] that the Walsh–Hadamard (WH) transform (or simply the Hadamard transform) [2] simplified the eigenstates of the quantum baker's map [3], a paradigmatic model of quantum chaos, considerably, the so-called 'Thue–Morse' state being a particularly good, even spectacular, example of this. Further, we showed that an exactly solvable permutation operator, viewed as the quantization of cyclic shifts, was useful in constructing a relevant basis [4]. We connect these two observations here and explicitly point out the rationale for the emergence of the Hadamard transform in the context of the quantum baker's map, and are therefore able to go beyond this by constructing a transform based on the Hadamard that simplifies the eigenstates of the quantum baker's map much more significantly. It is quite likely that the new transform will be useful in broader contexts since it combines the widely used discrete Fourier transform and the Hadamard transform in a novel manner.

We briefly review the relevant background, leaving some details that may be found in the references. The baker's map is a textbook example of a simple fully chaotic system.

The classical baker's map [5], $T$, is the area preserving transformation of the unit square $[0, 1) \times [0, 1)$ onto itself, which takes a phase space point $(q, p)$ to $(q', p')$ where

$$(q', p') = \begin{cases} (2q, p/2) & \text{if } 0 \leqslant q < 1/2 \\ (2q-1, (p+1)/2) & \text{if } 1/2 \leqslant q < 1. \end{cases} \tag{1}$$

The repeated action of $T$ on the unit square leaves the phase space completely mixed, this is well known to be a fully chaotic system that is Bernoulli. The area-preserving property makes this map a model of chaotic two-degree of freedom Hamiltonian systems, and the Lyapunov exponent is log(2) per iteration.

The classical baker's map is exactly solvable in many ways, including an explicit prescription for finding periodic orbits of any period, which follows on using the binary representation for the phase-space variables. The action of $T$ as a complete left-shift becomes clear in this representation. Due to its simplicity, and due to the fact that there is nothing but chaos (it is not a perturbation of an integrable system), its quantization due to Balazs and Voros [3] has been used extensively in studies of quantum chaos and semiclassical methods. It has also been experimentally implemented recently using NMR [6].

The 'usual' quantum baker's map, in the position representation is:

$$B = G_N^{-1} \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix}, \tag{2}$$

where

$$(G_N)_{nm} = \frac{1}{\sqrt{N}} \exp\left(\frac{-2\pi \mathrm{i}}{N}(n+1/2)(m+1/2)\right).$$

We require that $N$ be an even integer; Saraceno [3] imposed anti-periodic boundary conditions that we use. The Hilbert space is finite dimensional, the dimensionality $N$ being the scaled inverse Planck constant ($N = 1/h$), where we have used that the phase-space area is unity. The position and momentum states are denoted as $|q_n\rangle$ and $|p_m\rangle$ respectively, where $m, n = 0, \ldots, N-1$ and the transformation function $\langle q_n|p_m\rangle$ between these bases is the finite Fourier transform $(G_N)_{nm}$ given above.

The choice of anti-periodic boundary conditions fully preserves parity symmetry, here called $R$, which is such that $R|q_n\rangle = |q_{N-n-1}\rangle$. Classically this is the symmetry $(q \to 1-q, p \to 1-p)$. Time-reversal symmetry is also present and implies in the context of the quantum baker's map that an overall phase can be chosen such that the momentum and position representations are complex conjugates: $G_N\phi = \phi^*$, if $\phi$ is an eigenstate in the position basis. $B$ is a unitary matrix, whose repeated application is the quantum version of the full left-shift of classical chaos. Unlike the quantum cat map, where many analytical results concerning the spectrum are known [7], this is not the case with $B$. However, the quantum map $B$ is more generic than the quantum cat map, and hence its spectrum, especially the eigenstates, is of considerable interest.

We had shown that a simple exactly solvable shift operator $S$ acts as a good zeroth-order operator for the quantum baker's map, although there is no classical integrable 'zeroth order' system for the baker's map [4]. Here we also showed how one can construct a quantum baker's map $B_S$ that is very closely allied to $B$ above, using $S$, thereby explaining this closeness. It may therefore be expected that the eigenstates of $S$ form a basis in which the eigenstates of the quantum baker's map appear simple. Thus we seek to diagonalize the shift operator and use this as a basis to expand the eigenstates of the quantum baker's map.

However the spectrum of $S$ is typically highly degenerate, especially when $N$ is a power of 2; therefore we have multiple choices for the eigenstates. In [4] we had used $N$ that lead to the smallest degeneracy possible. Specifically those $N$ whose order modulo 2 is the largest possible value, $N-2$, have a nondegenerate spectrum saved for two states with unit eigenvalues. The case when $N$ is a power of 2 is however very interesting, as the eigenfunctions are well simplified using the Walsh–Hadamard transform, and there are connections to automatic sequences such as the Thue–Morse sequence [1]. Therefore it is natural that we seek to understand this case from the point of view that uses the shift operator essentially. To do this, we make use of the symmetries of the shift operator especially the parity and time-reversal to 'reduce' the eigenstates. The use of quantum symmetries is of course natural, and we note that in the case of quantum cat maps [7] the complete use of all quantum symmetries results in exactly solvable states that are ergodic and these have been called Hecke eigenfunctions [8].

## 2. Eigenfunctions of the shift operator

We will henceforth denote the position eigenkets $|q_n\rangle$ simply as $|n\rangle$ and use this as a basis unless otherwise stated. The shift operator $S$, by definition, acts on the position basis as

$$S|n\rangle = \begin{cases} |2n\rangle & 0 \leqslant n < N/2 \\ |2n - N + 1\rangle & N/2 \leqslant n \leqslant N-1. \end{cases} \tag{3}$$

The shift operator $S$ is a generalization of what was proposed as the quantum baker's map by Penrose [9] for the case when $N = 2^K$, $K$ integer. In this case, which is of sole concern in this paper, the Hilbert space is isomorphic to that of $K$ qubits, or two-level systems. Let $n = a_{K-1}a_{K-2}\cdots a_0$ be the binary expansion of $n$, so that

$$|n\rangle = |a_{K-1}\rangle \otimes |a_{K-2}\rangle \otimes \cdots |a_0\rangle, \tag{4}$$

where now the 'qubit' states $|0\rangle$ and $|1\rangle$ are orthonormal basis states. In the standard representation

$$|0\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5}$$

Of course the state $|0\rangle$ is not to be confused with the state $|n=0\rangle$ which is actually $\otimes^K|0\rangle$. In the following we will not work with individual qubits states as such for this confusion to arise.

The action of the shift operator is then transparent:

$$S|n\rangle = |a_{K-2}\rangle \otimes |a_{K-3}\rangle \cdots |a_0\rangle \otimes |a_{K-1}\rangle. \tag{6}$$

Thus the quantum operator $S$ embodies the left-shift action, by cyclically shifting the states from one qubit to its left 'neighbour'. It however re-injects the most significant bit at the least significant position, due to periodic boundary conditions, which ultimately naturally leads to periodicity. It is also helpful to think of $S$ as acting on the space $\{0,1\}^K$ consisting of binary strings of length $K$, which we denote generically by $\sigma$: $S(a_{K-1}a_{K-2}\cdots a_0) = a_{K-2}a_{K-3}\cdots a_0 a_{K-1}$.

The parity operator $R$ introduced earlier, can be written as a pure $K$-fold tensor product on this space, so that it can be thought of as local action on the individual qubits,

$$R = \otimes^K \mathcal{R}, \qquad \text{where} \qquad \mathcal{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

Thus the action on individual qubits is that of a flip. When considered as action on binary strings, $R(a_{K-1}a_{K-2}\cdots a_0) = \overline{a}_{K-1}\overline{a}_{K-2}\cdots\overline{a}_0$, where $\overline{0} = 1$ and $\overline{1} = 0$ are bit-flips. It is easy to see that $S$ commutes with $R$. While there are uncountably many operators that commute with $S$ (any operator of the form $\otimes^K \mathcal{A}$, where $\mathcal{A}$ is a 1-qubit operator, will commute with $S$), $R$ also commutes with the usual quantum baker's map $B$, and is thus an important symmetry for constructing a basis that is close to that of the eigenfunctions of the quantum baker's map. The classical limit of the unitary operator $S$ has been discussed earlier [4]. It can also be thought of as quantizing a multivalued mapping that is hyperbolic [4]; similar interpretations have been proposed for toy models of open bakers in [10]. However we continue to use it here only in so far as it enables us to understand the eigenstates of $B$.

It is particularly simple to diagonalize $S$ in the case when $N = 2^K$. Let $d$ be a divisor of $K$ (including 1 and $K$). Let $\sigma = s_1 s_2 \cdots s_d$ be a *primitive* binary string of length $d$, representing strings consisting of all its cyclic shifts. For example there are 12 primitive strings of length 4, consisting of three cycles. We represent these three cycles with the strings 0001, 0011 and 0111, chosen for convenience, to be the smallest when the cycle elements are evaluated as numbers. Let $\overline{\sigma}$ denote the string $\sigma$ repeated $K/d$ times and its value evaluated as a binary representation be $k$.

Then one set of eigenstates of $S$ constructed from these cycles is:

$$\left|\tilde{\phi}_l^k\right\rangle = \frac{1}{\sqrt{d}} \sum_{m=0}^{d-1} \mathrm{e}^{-2\pi \mathrm{i} l m/d} S^m |k\rangle, \qquad 0 \leqslant l \leqslant d-1 \tag{8}$$

and the corresponding eigenvalue is $\mathrm{e}^{2\pi \mathrm{i} l/d}$ [4]. If there are $p(d)$ such primitive representative strings of length of $d$, then $\sum_{d|K} d p(d) = 2^K$, where using the Mobiüs inversion formula, $p(n) = \sum_{k|n} \mu(n/k) 2^k/n$, ($\mu(n)$ is the Mobiüs function, $\mu(n) = 0$ if $n$ has a repeated prime in its prime factorization, otherwise it is $(-1)^r$, where $r$ is the number of primes in its factorization, and $\mu(1) = 1$.) Thus $\left|\tilde{\phi}_l^k\right\rangle$ form a complete and orthonormal set. Clearly there is degeneracy and this set is not unique. There is also freedom in the choice of the representative string $\sigma$; we will choose this to be the smallest integer when treated as a binary expansion. The eigenvectors arranged in columns of a matrix constitute a unitary transform which consists of direct sums of $p(d)$ discrete Fourier transforms (DFTs) of dimension $d$ each. It may be written as:

$$T_f = \bigoplus_{d|K} \bigoplus_{p(d)} F_d \tag{9}$$

where $F_d$ is a DFT of dimension $d$. For example if $K = 3$, the divisors are only 1 and 3, there are two subspaces of dimension 1 and two of dimension 3 corresponding to the cycles $\overline{0}, \overline{1}, \overline{001}, \overline{011}$. If the basis is arranged in the order 000, 001, 010, 100, 011, 110, 101, 111, we have chosen the first member of the cycle to be the smallest; the others are obtained by consecutive left-shifts, $T_f = F_1 \bigoplus F_3 \bigoplus F_3 \bigoplus F_1$.

### *2.1. Simultaneous eigenstates of parity and the shift operator*

The vectors $\left|\tilde{\phi}_l^k\right\rangle$ are however not eigenstates of parity $R$. In order to construct this we find a unitary operator $H$ of the form $\otimes^K \mathcal{H}$, so that it commutes with $S$, and such that $\mathcal{H}$ diagonalizes $\mathcal{R}$. This fixes

$$\mathcal{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{10}$$

and $H$ as the Walsh–Hadamard transform. It follows that

$$RH = Ht, \qquad t = \mathrm{diag}(1, -1, -1, 1, \ldots), \tag{11}$$

$t$ is a diagonal matrix whose entries are the Thue–Morse sequence [11]. Its $n$th term is $t_n = (-1)^r$, where $r = \sum_j a_j$ and $a_j$ are the bits in the binary expansion of $n$. They satisfy the iterative rule

$$t_{2n} = t_n, \qquad t_{2n+1} = -t_n, \qquad t_0 = 1. \tag{12}$$

Stated otherwise, the columns of the Walsh–Hadamard matrix $H$ have parities that are arranged according to the Thue–Morse sequence.

Consider the orthonormal complete set $|\phi_l^k\rangle = H|\tilde{\phi}_l^k\rangle$. Since $S$ and $H$ commute, this is clearly an eigenstate of the shift $S$. That it is also a parity eigenstate follows from:

$$R|\phi_l^k\rangle = \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1} \mathrm{e}^{-2\pi \mathrm{i} lm/d} S^m R H|k\rangle = \frac{1}{\sqrt{d}}\sum_{m=0}^{d-1} \mathrm{e}^{-2\pi \mathrm{i} lm/d} S^m H t|k\rangle = t_k|\phi_l^k\rangle. \tag{13}$$

where we have used that $R$ commutes with $S$ and $t_k$ is the $k$th member of the Thue–Morse sequence. Thus

$$S|\phi_l^k\rangle = \mathrm{e}^{2\pi \mathrm{i} l/d}|\phi_l^k\rangle, \qquad R|\phi_l^k\rangle = t_k|\phi_l^k\rangle. \tag{14}$$

We will also presently adapt these eigenstates to 'time-reversal'; however, before that we note that the plain Walsh–Hadamard transform shares some common rows with $\langle\phi_l^k|n\rangle$. For instance when $k = 0$ ($\sigma = (0)$) and when $k = N-1$ ($\sigma = (1)$) the rows $\langle\phi_0^0|n\rangle$ and $\langle\phi_0^{N-1}|n\rangle$ consisting of all ones, and the row with the Thue–Morse sequence respectively are common to the Hadamard matrix. Indeed since the Thue–Morse and closely allied sequence dominate the eigenfunctions of the quantum baker's map, the Hadamard transform works well in this context.

To be more explicit about the proposed transform we again illustrate with the case $K = 3$. Using the ordering described above, the structure of the transform is

$$\frac{1}{\sqrt{8}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}$$
$$\times \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{3} & & & & & & & \\ & 1 & 1 & 1 & & & & \\ & 1 & \omega & \omega^2 & & & & \\ & 1 & \omega^2 & \omega & & & & \\ & & & & 1 & 1 & 1 & \\ & & & & 1 & \omega & \omega^2 & \\ & & & & 1 & \omega^2 & \omega & \\ & & & & & & & \sqrt{3} \end{pmatrix}, \tag{15}$$

where $\omega = \mathrm{e}^{2\pi \mathrm{i}/3}$. The second matrix is what we called $T_f$, the first is essentially the Walsh–Hadamard matrix, but for the rearrangement of the columns according to cycles. Thus this transform which appears to be new, combines the DFT and the Hadamard transforms in an interesting way.

### 2.2. Time-reversal adapted states of the shift operator

The time-reversal symmetry of the quantum baker's map [3] is not the same as that of the shift $S$ as $G_N S G_N^{-1} \neq S^{-1}$. Therefore we cannot have the eigenstates of $S$ to be such that its Fourier transform is identical to its complex conjugate. However there is an analogous symmetry:

$$\mathsf{b}_0 S \mathsf{b}_0 = S^{-1}, \qquad \mathsf{b}_0 |a_{K-1} a_{K-2} \cdots a_0\rangle = |a_0 a_1 \cdots a_{K-1}\rangle. \tag{16}$$

Here $\mathsf{b}_0$ is the bit-reversal operator; it reverses the significance of the bits in a binary string. Clearly $\mathsf{b}_0^2 = 1$. Its emergence is linked to the fact that the periodic points of the baker's map are such that the position and momentum are bit-reversals of each other. Its connection to the Fourier transform is made even more closer in the context of the baker's map when we note that

$$B = \left(G_N^{-1}\right)_0 (\mathbb{1} \otimes (G_{N/2})_1) \tag{17}$$

where the additional subscripts on the Fourier transform refer to the number of most significant bits that are left out while performing the transform, which is therefore a 'partial Fourier transform' as defined by [12]. In particular $(G_N)_0$ is the full transform of all the qubits and is therefore of dimensionality $N = 2^K$, while $(G_{N/2})_1$ transforms the $K-1$ least significant bits and has the dimensionality $N/2 = 2^{K-1}$; the first qubit is left unaltered. This is identical to the usual quantum baker's map in equation (2). Analogously, it is easy to see by acting on bit strings that

$$S = \mathsf{b}_0 (\mathbb{1} \otimes \mathsf{b}_1) \tag{18}$$

where $\mathsf{b}_k$ bit reverses the $L-k$ least insignificant bits, for example $\mathsf{b}_0$ reverses the whole string.

The time-reversal symmetry of $S$ then implies that eigenstates $|\psi\rangle$ of $S$ may be chosen such that $\mathsf{b}_0|\psi\rangle = |\psi^*\rangle$, where the complex conjugation is done in the standard position basis. Now the state $\left|\phi_l^k\right\rangle$ need not be of this kind; therefore, it is necessary to multiply by suitable phases or take appropriate linear combinations of them. Towards this end we define two bit strings $\sigma$ and $\sigma'$ shift-equivalent ($\sigma \sim \sigma'$) if one is the result of repeatedly applying the cyclic shift operator $S$ to the other. There are two kinds of binary strings; we label them type-A and type-B. If $\sigma$ is a string of type-A then $\mathsf{b}_0\sigma \sim \sigma$, and it is otherwise of type-B. It is somewhat surprising that the smallest binary string of type-B is of length 6 and that there are only 2 of them: 110010, 110100, apart from their cyclic shifts. It is easy to see that if $\sigma$ is of type-B then $S^m\sigma$ is also of type-B for any integer $m$.

If $\sigma$ (value $k$) is a string of type-A then there exists an integer $p$ such that $S^p\sigma = \mathsf{b}_0\sigma$. Also we note that $\mathsf{b}_0$ commutes with the Hadamard matrix $H$. Using these facts, a short calculation shows that if

$$\left|\psi_l^k\right\rangle = \mathrm{e}^{\mathrm{i}\pi l p/d}\left|\phi_l^k\right\rangle, \qquad \text{then} \qquad \mathsf{b}_0\left|\psi_l^k\right\rangle = \left|\psi_l^{k*}\right\rangle. \tag{19}$$

Thus type-A strings give rise to states that are time-reversal adapted up to multiplication by a phase. If $\sigma$ is of type-B, and its bit-reversal $\mathsf{b}_0\sigma$ evaluates to $k'$, we note that $\mathsf{b}_0\left|\phi_l^k\right\rangle = \left|\phi_l^{k'*}\right\rangle$ and $\mathsf{b}_0\left|\phi_l^{k'}\right\rangle = \left|\phi_l^{k*}\right\rangle$. Therefore we can construct the linear combinations:

$$\left|\psi_l^{k\pm}\right\rangle = \frac{\mathrm{e}^{-\mathrm{i}\alpha_\pm}}{\sqrt{2}}\left(\left|\phi_l^k\right\rangle \pm \left|\phi_l^{k'}\right\rangle\right), \tag{20}$$

where $\alpha_+ = 0$ and $\alpha_- = \pi/2$, which are orthogonal and such that $\mathsf{b}_0\left|\psi_l^{k\pm}\right\rangle = \left|\psi_l^{k\pm*}\right\rangle$. Thus this way we can construct a transform whose elements are $\langle\psi_j|n\rangle$, where $j$ labels all the parity and time-reversal adapted states of the shift-operator $S$. In practice we order binary strings in the following way: 0, 1, 10, 100, 110, 1000, 1100, 1110, 10000, 10100, ..., such that if $\sigma$ and $\sigma'$

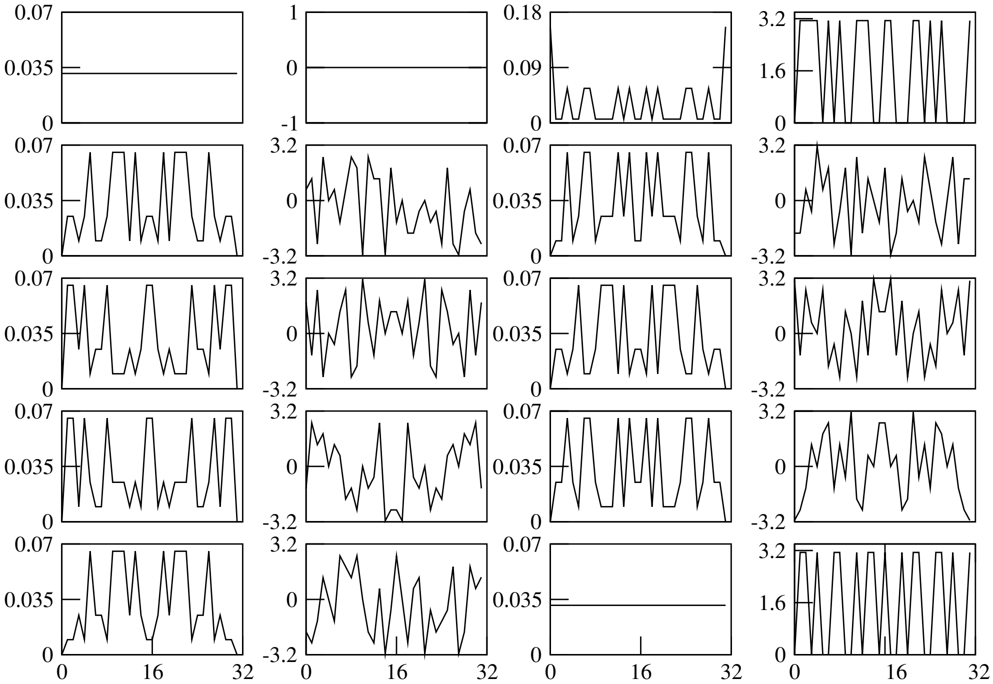


**Figure 1.** Ten basis states $|\psi\rangle$ that are simultaneous eigenfunctions of the shift operator, parity (bit-flip) and time-reversal (bit-reversal) for the case $N = 32$. Shown in the first and third columns are the intensities ($|\langle n|\psi\rangle|^2$) while the second and fourth columns have the corresponding phases. The first state is the uniform state, while the last has the Thue–Morse sequence as the components.

are two members on the list then $\sigma$ or $\mathsf{b}_0\sigma$ are not shift-equivalent to $\sigma'$. Among the possible representatives we choose that which is *largest* when treated as a binary representation of an integer. Choosing the largest, as opposed to the smallest as in the previous case, gives us a unique increasing sequence that appears to be new. Given any $N = 2^K$, we choose from this list strings whose length are divisors of $K$. These would include strings of both type-A and type-B; in either case we construct a set of states based on the algorithm outlined above, finding $p$ by inspection in the case of type-A strings.

In figure 1 we show ten of these states for the case $N = 32$, showing the simplest uniform state, the Thue–Morse sequence and eight other states that involve appropriate combinations of Fourier transforms of the columns of the Hadamard matrix.

## 3. The eigenstates of the quantum baker's map in the new basis

If $|\Phi_k\rangle$ is an eigenvector of $B$ then we refer to its representation in the position basis as basis-0 ($\langle n|\Phi_k\rangle$), in the Hadamard basis as basis-1 ($\langle n|H|\Phi_k\rangle$), in the parity adapted basis of the shift operator as basis-2 $\left(\left\langle \phi_l^\sigma \middle| \Phi_k\right\rangle\right)$, in the parity and time-reversal adapted basis of the shift operator as basis-3 ($\langle\psi_j|\Phi_k\rangle$).

In figure 2 we show ten most simplified eigenfunctions for $N = 2^9 = 512$ using the new representations. For comparison we show the states in both basis-0 (position) and basis-2. We found that for these states basis-3 did not yield further significant simplifications; we will presently quantify the degree of simplification across the spectrum. The state that is most

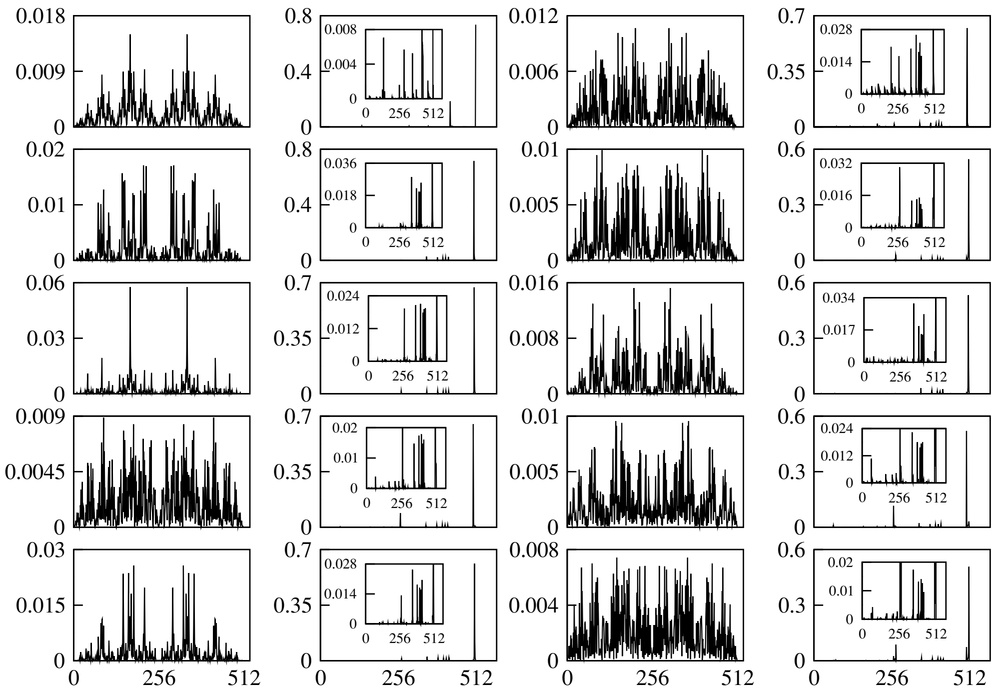

**Figure 2.** Ten eigenstates of the quantum baker's map for $N = 512$ that are most simplified on using basis-2 or basis-3. Shown in columns 1 and 3 are the intensities of the states in the position basis. Columns 2 and 4 show the corresponding intensities on using basis-2 (along with the 'grass' in the insets), this basis having states that are simultaneous eigenstates of the shift operator and parity. The Thue–Morse state is the first one.

simplified continues to be the Thue–Morse state which we have discussed in an earlier work. Its transformation whether using the Hadamard or the new transformations continues to be dominated by the Thue–Morse sequence $t_k$. As we have pointed out earlier this state is well described by the ansatz $t_K + G_N t_K$ [1], where $t_K$ is the normalized Thue–Morse sequence of length $2^K = N$ treated as a vector. In other words, the Thue–Morse sequence and its Fourier transform dominate the state. The Fourier transform of the Thue–Morse sequence is known to be a multifractal [13] and we established that the quantum eigenstate we called the Thue–Morse state was also a multifractal [1]. The structure of the eigenstate is dominated by the Fourier transform of the Thue–Morse sequence, the peaks being much larger than $1/\sqrt{N}$. The peaks occur at the period-2 orbit of the doubling map (at 1/3, 2/3) and at the points corresponding to homoclinic orbits of this point. Since the Thue–Morse states form a sequence for increasing $N$, and $N \to \infty$ is the classical limit, it may be expected that the Thue–Morse state is related to some classical invariant measure of the classical baker's map. We now show that this is indeed the case.

Since we are dealing with the position representation of the states, and since the baker's map is such that its position coordinate evolves (in the forward direction) independently of the momentum according to the doubling map ($x \mapsto 2x(\text{mod}1)$), the invariant density is simply that of this one-dimensional map. If $\rho(x)$ is an invariant density of this map, we must have that:

$$\rho(x) = \frac{1}{2}\rho\left(\frac{x}{2}\right) + \frac{1}{2}\rho\left(\frac{x+1}{2}\right). \tag{21}$$

If $\rho(x) = \sum_{k=0}^{\infty} t_k \exp(2\pi i k x)$ then it is easy to see, using the recursion relation we have stated earlier namely $t_{2k} = t_k$, that it satisfies the requirement of an invariant density. Since the Thue–Morse state is close to the Fourier transform of the Thue–Morse sequence, it is suggestive that $\rho(x)$ is relevant to the classical limit of such states. Thus we believe we have a concrete example of a set of states of a quantum chaotic system that limits to a classical invariant measure that is not the ergodic measure, which would be uniform in this case. Instead, it is a multifractal measure that is strongly peaked at period-2 periodic orbits and all their homoclinic excursions. This is opposed to the known examples where the limit is either ergodic or has delta-peaks corresponding to classical periodic orbits [14]. While we have given evidence of this without establishing it rigorously, this is an interesting deviation from quantum ergodicity that is allowed by Schnirelman's theorem [16]. It is very likely that many other eigenstates of the quantum baker's map are also of this kind, limiting to classical invariant measures that are non-ergodic and are multifractal. That the eigenfunctions are multifractal we have already indicated in an earlier work [1].

It is pertinent here to connect with the works of Nonennmacher and co-workers, who have studied the quantum cat maps [14] and the Walsh-quantized baker's map [15], which is an exactly solvable toy model of the baker's map. Among other things they have found two types of states in the Walsh-quantized baker's map, the first which they call 'half-scarred' has in the semiclassical limit part of its measure on classical periodic orbits and part is equidistributed in the Lebesgue measure. Such states have also been constructed by them for the quantum cat maps [14]. The tensor-product states found for the Walsh-quantized baker's map [15] have semiclassical measures that are singular Bernoulli measures and were constructed as tensor products of states of the underlying 'qubit' space (when $N$ is a power of 2 as in this paper, but have been generalized to other powers). The Thue–Morse states of the quantum baker's map ('Weyl' quantized) under discussion seem to be closer to the tensor-product states than the 'half-scarred' ones. For one, there is strong evidence that the measure is multifractal in the semiclassical limit, and also the scarring can be unambiguously associated with short period periodic orbits and homoclinic ones [1]. Moreover, the Thue–Morse state is 'close' to the simple tensor product $\otimes^K(|0\rangle - |1\rangle)/\sqrt{2}$ for $N = 2^K$, which indeed is the finite Thue–Morse sequence. This can be measured in terms of the modulus of the inner product and while this does decrease with $N$, it does so slowly. Numerical calculations not shown here indicate a decay of $N^{-0.1}$. Note that typical inner products with random states will scale as $N^{-0.5}$. A more accurate representation of the Thue–Morse state as a superposition of the above product and its Fourier transform is such that their inner product decays even more slowly to zero (as $N^{-0.08}$). We postpone a more detailed description to a later publication, suffice to say that the resolution of the semiclassical measure of the Thue–Morse state and in general other states of the quantum baker's map into singular-continuous, pure-point and continuous components is not yet clear.

To quantify the extent to which the eigenvectors $|\Phi_k\rangle$ of $B$ are simplified we evaluate the participation ratios (PR). When a complete orthonormal basis $\{|\alpha_i\rangle, i = 0, \ldots, N-1\}$ is used the PR is defined as

$$\left(\sum_{i=0}^{N-1} |\langle \alpha_i|\Phi_k\rangle|^4\right)^{-1}. \tag{22}$$

The PR is an estimate of the number of $|\alpha\rangle$ basis states needed to construct the vector $|\Phi_k\rangle$, here chosen to be one of the eigenstates of $B$. We calculate the PR in (1) the position basis (basis-0, $|\alpha\rangle = |n\rangle$), (2) the Hadamard basis (basis-1, $|\alpha\rangle = H|n\rangle$), (3) the basis that consists of parity reduced eigenstates of $S$ (basis-2, $|\alpha\rangle = |\phi_l^k\rangle$), and (4) the basis that has both parity and time-reversal symmetry reduced eigenstates of $S$ (basis-3, $|\alpha\rangle = |\psi_j\rangle$).

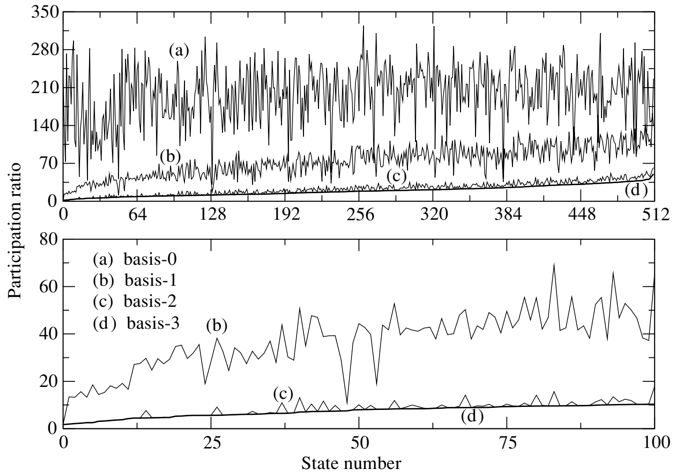


**Figure 3.** The participation ratio of the eigenstates of the baker's map in the position basis (basis-0), the Walsh–Hadamard basis (basis-1), the basis that is parity reduced eigenstates of $S$ (basis-2), and in the basis that is also time-reversal symmetry adapted (basis-3). The states are arranged in the increasing order of the PR in basis-3. This is for the case $N = 512$, and the lower figure is a magnification of details for the first hundred states.

From figure 3 we indeed see that the eigenstates of $S$, properly symmetry reduced simplify the eigenstates of the quantum baker's map $B$ significantly. This does considerably better than the previously used Walsh–Hadamard transform. The difference is only however marginal for the Thue–Morse state, as the last column of the Walsh–Hadamard transform, the Thue–Morse sequence of finite generation, is anyway a parity and time-reversal symmetric eigenstate of the shift operator $S$. We see from the figure that for $N = 512$, there are about hundred states that can be constructed from ten or fewer eigenstates of the shift operator as we have constructed them above. We also note that while there is considerable difference between using the basis-2 and the Hadamard basis, basis-1, there is not that much simplification due to the use of basis-3 over basis-2. This is understandable as parity symmetry ($R$) is the same for both $S$ and $B$, while the time-reversal symmetries are different.

Due to the simplicity and efficacy of basis-2, we will discuss this further. We first point out that the dual or momentum basis $G_N|\phi_l^k\rangle$ is exactly as effective as the original basis for studying the eigenfunctions of the baker's map, due to time-reversal symmetry. To prove this note that

$$G_N|\phi_l^k\rangle = -G_N^{-1}R|\phi_l^k\rangle = -t_k G_N^{-1}|\phi_l^k\rangle \tag{23}$$

where we have used $G_N^2 = -R$ (for example, see Saraceno in [3]), and equation (14). Using time-reversal of the eigenstates of $B(G_N|\Phi\rangle = |\Phi^*\rangle)$ implies that

$$\langle\Phi|G_N^{-1}|\phi_l^k\rangle = \left(\sum_n \langle\Phi|n\rangle\langle n|\phi_l^k\rangle^*\right)^* = \langle\Phi|\phi_{l'}^k\rangle^* \tag{24}$$

where $l' = d - l$ unless $l = 0$, in which case $l' = 0$ as well. Hence finally

$$\langle\Phi|G_N|\phi_l^k\rangle = -t_k\langle\Phi|\phi_{l'}^k\rangle^*. \tag{25}$$

Thus while the overlaps of the eigenstates of the quantum baker's map in a basis which is the the Fourier transform of the basis-2 are not the same as originally, they are upto a sign, complex conjugates of overlaps with some other basis states with a different value of $l$ in general. Clearly this leads to identical participation ratios. Thus the Fourier transform of basis-2 is also of interest, indeed as we have already indicated, many of the eigenstates of the baker's map look like them, the foremost being the Thue–Morse state where the momentum representation of $\left|\phi_0^{(N-1)}\right\rangle$ is of relevance. Thus we have a natural way of generalizing this class of functions. We do not pursue this further here, but note that the other Fourier transforms are also of relevance to the spectrum of the quantum baker's map, and they also have multifractal characters. A few related functions have recently been studied by us as the 'Fourier transform of the Hadamard transform' [17]. It is also reasonable to expect that a combination of basis-2 and its Fourier transform maybe even closer to the actual eigenstates of the quantum baker's map than even basis-2. For instance in the case of the Thue–Morse state, such a combination does better than either the Thue–Morse sequence or its Fourier transform taken individually [1].

To summarize, in this work we have constructed eigenfunctions of the shift operator that have additional symmetries of bit-flip or parity and bit-reversal or time-reversal. Using these we have seen why the Walsh–Hadamard transform simplifies states of the quantum baker's map, as well as shown that these transforms are capable of doing significantly better. The use of these transforms in other contexts, other than the quantum baker's map, is possible. It combines elements of both the Fourier and the Hadamard transforms in an interesting way. Using these transforms helps us study the eigenfunctions of the baker's map in a more detailed manner, and our future work will explore this further. Operators akin to the shift operator have been used as toy models of open quantum bakers to study fractal Weyl laws [10]. It is also interesting that the same cocktail of the shift operator, the Fourier transform and the Hadamard transform appears essentially in Shor's quantum algorithm for factoring, a fact also previously pointed out in [4]. A very recent work has made a significant contribution by constructing suitable basis sets for $N$ that are not powers of 2 [18].

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